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**EXPLICIT SUPERSTRING VACUA IN A BACKGROUND OF  
GRAVITATIONAL WAVES AND DILATON**

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**Abstract**

We present an explicit solution of superstring effective equations, represented by gravitational waves and dilaton backgrounds. Particular solutions will be examined in a forthcoming note.

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# 1 Introduction

In the conclusions of a recent paper on supersymmetric non-linear sigma models with Brinkmann metric [1], we announced the possibility to display explicit metric and dilaton backgrounds which satisfy the Weyl conditions. The assumption, for this result to obtain, was that the transverse space of the model be hyperKähler, therefore implying for this model an  $N = 4$  supersymmetry. Such supersymmetric model has been worked out explicitly. It provides the backgrounds for both the metric and the dilaton in closed form and is the purpose of this short communication.

After a brief outline of the hyperKähler manifold chosen for the transverse space of the model, we give the metric in this space, that was taken in the Calabi series [2], i.e. a cotangent bundle over  $CP^1$ :  $T(CP^1)^*$ . The background for this model is notoriously self-dual: of the Eguchi-Hanson type. At this point the crucial feature is that  $N = 4$  supersymmetric sigma-models (requiring a manifold which is hyperKähler), are known [3] to have an identically zero beta-function<sup>1</sup>. Accordingly the Weyl invariant background for the metric is equal, up to a multiplicative constant<sup>2</sup>, to the original metric one started with. Moreover, the generic reparametrization term  $M$  reduces to  $M\mu = \partial_\mu\phi$  since  $W\mu$  can be shown to be zero for such models having vanishing beta-function of their transverse part.

Finally, as these hyperKähler spaces are necessarily Ricci-flat [4], the dilaton may be reduced to its linear form in the  $0^{\text{th}}$  and  $(n + 1)^{\text{th}}$  coordinates of the model, i.e. the light cone coordinates  $u$  and  $v$ .

As conclusions and perspectives, we call the reader's attention on the following fact: metric on various hyperKähler manifolds are seldom explicitly known, although, for most of them, existence theorems can be proven (for instance the celebrated K3-spaces). Recent studies, however, have considerably enlarged the field [5]. The progresses in this direction should allow to formulate whole series of valuable models with non trivial space-time. The existence of an  $N = 4$  world sheet superconformal symmetry seems to have a stabilizing effect on the perturbative solution. Actually we refer to the advances made in [5] on superstrings in wormhole-like backgrounds.

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<sup>1</sup>Notice that preservation of  $N = 4$  supersymmetry in perturbation theory requires that the metric plus eventual counterterms preserve the Ricci-flatness [4].

<sup>2</sup>In principle the sigma-model inverse constant  $f$  might be a complicated function of the  $(n + 1)^{\text{th}}$  coordinate  $u$ . Here it is constant.

The basic features of the supersymmetric non-linear sigma-model with Brinkmann metric, Minkowski signature and covariantly constant null Killing vector, will of course not be recalled here, as a series of rather exhausting papers on the subject have appeared these two last years [6]–[9]. We recall the general form of the line element for a suitable choice of coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -2du dv + g_{ij}(x, u) dx^i dx^j \quad (1)$$

$\mu, \nu = 0, 1, \dots, n, n+1$ ;  $i, j = 1, 2, \dots, n$  (real indices). The transverse part (with latin indices) has a metric that we will choose in such a way as it complies with the manifold described above on which transverse strengths take values. We will start, to alleviate the computation by assuming that  $g_{ij}(x, u)$  can be brought to the form

$$g_{ij}(x, u) = f(u) g_{ij}(x) \quad (2)$$

as was done in [6]–[9]. This by no means prohibits a possible but more involved treatment of the problem by invoking further additional symmetries which we shall not need to take into account here. The function  $f(u)$  in (2) can be shown to be the inverse coupling of the supersymmetric transverse sigma-model and is therefore defined for a family of theories with various values of  $u$ . It can be shown that  $f(u)$  is running with  $u$  and will satisfy a RG-like equation

$$p\dot{f}(u) = \beta f \quad (3)$$

with  $p$  a constant and  $\beta(f)$  the beta-function of the transverse sigma-model, defined by the transverse  $\beta G_{ij}$

$$\beta_{ij}^G = \beta(f) \gamma_{ij} . \quad (4)$$

Finiteness of the model on a flat 2d background requires in addition that the  $n+2$  sigma-model with target space metric  $g_{\mu\nu}$  beta-function has to vanish up to a reparametrization term  $D_\mu M_\nu$ .  $M_\mu$  is not arbitrary, and to establish Weyl invariance of the model, the existence of an adequate dilaton background  $\phi$  must be proved, such that  $M_\mu$  is represented by

$$M_\mu = \alpha' \partial_\mu \phi + 1/2 W_\mu . \quad (5)$$

## 2 The explicit model

The origin of  $W_\mu$  has been discussed in several papers (see [6]–[9] and [11] for instance) and we do not repeat here the information we have on it. Similarly we do not repeat either the  $N = 1$  supersymmetric extension [8, 9] of the model with bosonic action [8, 9].

In previous works on supersymmetric models of the kind studied in this note, the number of supersymmetries considered for the transverse part was  $N = 2$ . Accordingly, Kähler manifolds were considered and especially symmetric and homogeneous Kähler manifolds. These had the properties that the beta-function in (3) reduced to a constant and, by integration,  $f(u)$  was a constant times  $u$ , i.e.

$$f(u) = bu$$

(Kähler transverse space, homogeneous and symmetric).

In the present note on the same class of models, we assume the transverse space to be an hyperKähler manifold. We will suppose the reader somewhat familiar with differential geometry and Lie groups in order to avoid to discuss here the holonomy group of such manifolds, in particular the absence of a  $U(1)$  factor in the holonomy group<sup>3</sup> being the signature of a Ricci-flat manifold. Also we suppose the reader acquainted with the Clifford algebra fulfilled by their three complex structures. Anyhow, one of the main features that must be kept in mind is that hyperKähler manifolds are Kählerian and quaternionic. For concreteness we specify the metric of our transverse space, as explained at the beginning of this note, to be that of a cotangent bundle over  $CP^1$ :  $T^*(CP^1)$ . The local hyperKähler structure of this space is proved by observing three linearly independent covariantly constant complex structures (or equivalently three closed 2-forms on  $T^*(M)$  and show their interrelation through the  $SU(2) \approx Sp(1)$  group transformations. This can be found in the literature [12]–[14].

Another feature to take into account is that, as can be seen by construction, a higher symmetry of the metric on  $T^*(M)$  is guaranteed or at least coinciding with the symmetry  $G$  of the initial manifold  $M$ , here  $CP^1$ . The latter is known to be a symmetric space isomorphic to  $SU(2)/U(1)$  (or  $SO(3)/SO(2)$ ). With our example ( $T^*(CP^1)$ ) we automatically specialize to a transverse space which is symmetric. Then we can appreciably simplify the

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<sup>3</sup>The generator of this  $U(1)$  factor is the Ricci-form  $R_{\alpha\bar{\gamma}\delta}^\alpha dz^\gamma \wedge d\bar{z}^\delta$  (complex indices).

analysis<sup>4</sup>. In particular, there are both: 1) restrictions on the form of the dilaton potential which must be independent of the transverse space coordinates and can be written as

$$\phi = pv + \phi(u) \quad (6)$$

and 2) non-renormalization theorems for the dilaton [10] which were already true at the level of  $N = 2$  supersymmetry (i.e. Kähler transverse spaces), studied in [8] and [9] and which can be shown to be a *fortiori* valid for hyperKähler spaces like the one appearing in the present note. This property considerably simplifies the Weyl invariance differential equation which was, in the  $N = 2$  case

$$\dot{\phi} = \frac{A}{p} f^{-1} \beta(f) - \frac{W_u}{4} + q; \quad (7)$$

$A, p, q$  constants,  $f$  constant, as  $\beta(f) = 0$  ( $q = 0$  for critical  $D = 10$ ).

We have explicitly shown in [9] that  $W_u$  was not vanishing on homogeneous Kähler spaces, due to the role of  $f(u) = bu$  in raising and lowering the indices by  $g_{ij}(x, u) = f(u)\gamma_{ij}(x)$  and its inverse, so that  $W_u$  is proportional to  $\partial_u S(u) \neq 0$ ,  $S(u)$  being the well-known globally defined trace of curvature power series (see also [15]). In the present case  $f$  is a constant and as such cannot introduce  $u$  dependence.  $W_u$ , like the other components of  $W$  is vanishing in the present instance. Furthermore the first term in (7) right-hand side is vanishing as  $\beta(f)$  does, and finally the answer for the dilaton backgrounds reads

$$\phi = \phi(u, v) = pv + qu + \phi_0 \quad (8)$$

( $\phi_0$  constant) as announced at the very beginning of this note.

What remains now is to display the line element in this model. As mentioned we took it as the simplest case ( $n = 1$ ) in the Calabi series. That means we have to display the metric for a cotangent bundle over  $S^2 = \mathbb{CP}^1$ . This metric exists in the literature at several places (e.g. [2]). Taking complex coordinates for convenience and short-hand notations, the metric on any  $T^*(M)$  has a block-form

$$g_{T^*M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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<sup>4</sup>Rigorously speaking, we have already implicitly used this symmetry property in writing (3) and (4), although parent relations can be written down for non symmetric spaces at the price of a much more involved analysis (for comments see, for example, Ref. [8]). Personally we did not try to examine this case at this moment.

with complex  $n \times n$  matrices  $A, B, C, D$ , expressed in particular through the Kähler metric on  $M$ :

$$g_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\bar{\beta}}K(z1, \bar{z}1) ; \quad (9)$$

( $z1^{\alpha}$  ( $\alpha = 1, \dots, n$ ), a  $n$  component complex variable)  $K(z1, \bar{z}1)$  being the Kähler potential.

One denotes another  $2n$  real coordinates in cotangent space through  $z2_{\alpha}(\bar{z}2_{\bar{\alpha}})$ , ( $\alpha = 1, \dots, n$ ). In our case, with  $M = \mathbb{CP}^1$ ,  $n = 1$ , we have a 4-dimensional cotangent bundle  $T^*(\mathbb{CP}^1)$  as the transverse space in (1). Out of these variables, the following scalar quantity can be defined

$$t = g^{\alpha\bar{\beta}}(z1, \bar{z}1)z2_{\alpha}\bar{z}2_{\bar{\beta}} \quad (10)$$

and the application

$$\Delta(t) = [-1 + (1 + 4t)^{1/2}]/2t \quad (11)$$

is a solution of the Ricci-flatness condition

$$\det(g_{T^*M}) = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{constant}$$

with adequate normalizations and regular behaviour at infinity. One must say that it is remarkable that the determinant does not generally depend on  $z1$  and  $z2$  separately, but well on  $t$  only. This feature allows one to deduce the solution (11). It is necessary for (11) to obtain. So, a candidate for a Kähler potential on  $T^*(\mathbb{CP}^1)$  can be guessed in the form

$$K_{T^*M}(z1, \bar{z}1, z2, \bar{z}2) = K(z1, \bar{z}1) + \mathcal{I}(t) \quad (12)$$

In doing the calculations of  $\det(g_{T^*M})$ , it appears soon that the candidate (12) is improper for most of the Kähler manifolds  $M$  in  $T^*(M)$ . But it appears also that in the Calabi series ( $M = \mathbb{CP}^n$ ), (12) is the right form to consider and in particular for  $T^*(\mathbb{CP}^1)$ ,

$$\Delta(t) = \mathcal{I}'(t) \equiv \frac{d\mathcal{I}(t)}{dt} . \quad (13)$$

After some algebra, we can write the line element for the transverse space in the following form (depending on  $g_{\alpha\bar{\beta}}$ , its inverse,  $\mathcal{I}(t)$ , and derivatives with respect to  $t$ , and on  $t$  itself of course:

$$ds_T^2 = A_{\alpha\bar{\beta}}dz1^{\alpha}d\bar{z}1^{\bar{\beta}} + B_{\alpha}^{\bar{\beta}}dz1^{\alpha}d\bar{z}2_{\bar{\beta}} + C_{\bar{\beta}}^{\alpha}d\bar{z}1^{\bar{\beta}}dz2_{\alpha} + D^{\alpha\bar{\beta}}dz2_{\alpha}d\bar{z}2_{\bar{\beta}} \quad (14)$$

with A, B, C and D given in appendix. Therefore (1) becomes

$$ds^2 = -2 du dv + f ds_T^2; \quad f \text{ constant} . \quad (15)$$

Hence, the resulting backgrounds (14), ((15) below) and (8) represent exact explicit solutions of superstring effective equations, the so-called fixed point direct product solutions. What is given below can be found at several places in the literature. It seemed to the author convenient for the reader to have this material close at hand.

### 3 Mathematical Appendix

1) A, B, C and D appearing in formula (13) of the text read

$$\begin{aligned} A_{\alpha\bar{\beta}} &= g_{\alpha\bar{\beta}} + \Delta \cdot R_{\alpha\bar{\beta}\gamma\bar{\delta}} z^\gamma \bar{z}^\delta + \Gamma_{\alpha\gamma}^\mu z^\mu D^{\gamma\bar{\delta}} \Gamma_{\bar{\beta}\bar{\delta}}^\nu \bar{z}^\nu \\ D^{\alpha\bar{\delta}} &= \Delta \cdot g^{\alpha\bar{\beta}} + \Delta' \cdot z^\alpha \bar{z}^\delta \\ B_{\alpha}^{\bar{\beta}} &= \Gamma_{\alpha\gamma}^\mu z^\mu D^{\gamma\bar{\beta}} \\ C_{\bar{\beta}}^\alpha &= D^{\alpha\bar{\delta}} \cdot \Gamma_{\bar{\beta}\bar{\delta}}^{\bar{\gamma}} \bar{z}^\gamma \\ \Delta &\equiv \Delta(t); \quad \Delta' \equiv \frac{d\Delta(t)}{dt} \end{aligned} \quad (16)$$

$g_{\alpha\bar{\beta}}$ ,  $g^{\alpha\bar{\beta}}$ ,  $\Gamma_{\beta\gamma}^\alpha$ ,  $R_{\beta\gamma\bar{\delta}}^\alpha$  are respectively the hermitian metric, its inverse, the connection and the curvature tensor on  $CP^n$ .  $\Delta$  is given by (11) in the text.

2) Complex conjugation:

$$\overline{A_{\alpha\bar{\beta}}} = A_{\bar{\alpha}\beta}; \quad \overline{D^{\alpha\bar{\beta}}} = D^{\bar{\alpha}\beta}; \quad \overline{B_{\alpha}^{\bar{\beta}}} = C_{\bar{\alpha}}^\beta$$

3) Conditions for the metric on  $T^*(CP^n)$  to be hyperKähler:

$$\begin{aligned} D^{\alpha\bar{\beta}} A_{\bar{\beta}\gamma} - C_{\bar{\beta}}^\alpha B_{\gamma}^{\bar{\beta}} &= \delta_{\gamma}^{\alpha} \\ C_{\bar{\beta}}^\alpha D^{\bar{\beta}\gamma} &= D^{\alpha\bar{\beta}} C_{\bar{\beta}}^{\gamma} \\ B_{\alpha}^{\bar{\beta}} A_{\bar{\beta}\gamma} &= A_{\alpha\bar{\beta}} B_{\gamma}^{\bar{\beta}} \end{aligned} \quad (17)$$

plus three analogous conditions, got from (16) by complex conjugation.

4)  $\text{CP}^1$  case.

The indices  $\alpha, \beta, \dots, \bar{\alpha}, \bar{\beta}, \dots$  can be dropped, as we have only two complex variables  $z_1, z_2$  and their complex conjugates  $\bar{z}_1, \bar{z}_2$ . Using the explicit values of the metric, the connection and curvature tensor for the  $M = \text{CP}^1$  case, the transverse line element can be cast in the simple form

$$ds_T^2 = G_{z_1\bar{z}_1}|dz_1|^2 + G_{z_1\bar{z}_2}d\bar{z}_1dz_2 + G_{\bar{z}_1z_2}dz_1d\bar{z}_2 + G_{\bar{z}_2z_2}|dz_2|^2 \quad (18)$$

with the  $G$ 's given by

$$\begin{aligned} G_{z_1\bar{z}_1} &= \frac{1 + 4t(1 + |z_1|^2)}{\sqrt{1 + 4t} \cdot (1 + |z_1|^2)^2} \\ G_{z_1\bar{z}_2} &= \bar{G}_{\bar{z}_1z_2} = \frac{2\bar{z}_1z_2(1 + |z_1|^2)}{\sqrt{1 + 4t}} \\ G_{\bar{z}_2z_2} &= \frac{(1 + |z_1|^2)^2}{\sqrt{1 + 4t}} \end{aligned} \quad (19)$$



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